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OPTIMIZATION OF THE STRUCTURE OF COMPOSITES*

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The problem of optimizing the electrical conducting properties of a two-phase composite is considered. The optimum geometry of the phase distribution of materials from which the composite is constructed, is sought. In some cases the best structure of the composite represents the limit of small-scale structures, in which the region occupied by the phases fragment into pieces, of size tending to zero. The idea of using, in such situations, the theory of averaging the equations with rapidly oscillating coefficients to construct the optimizing sequences of structures was suggested in /1/. Below the theory of averaging is used to solve the plane problem of the electric current in a ring filled with a two-phase composite, with fixed concentrations of phases. The small-scale geometry of the distribution of the composite phases is found for which the electrical resistance of the ring is lowest.

1. Formulation of the problem. We consider a two-phase composite occupying the region V in a plane. The region is split into two parts, V_+ and V_-, filled with electrically conducting materials with corresponding electrical conductivities σ_+ and σ_- (Fig.1). The values φ_1 and φ_2 $(\varphi_2 > \varphi_1)$ of the electric potential are given on two segments Σ_1 and Σ_2 of the boundary of the region. The remaining part of the boundary is electrically insulated. The value J of the current flowing through the region V depends on the geometry of the distribution of the regions V_+ and V_- in V, i.e. on the structure of the composite. We require to find the structure for which the current is largest or smallest. The areas of the regions V_+ and V_-, or in other words the concentrations of the composite phases, are assumed to be known.

If the structure of the composite is given, then the current J and electric potential $\varphi(x)$ can be found with help of the variational principle

$$(\varphi_2 - \varphi_1) J = \inf \langle \sigma(x) (\nabla\varphi(x))^2 \rangle \tag{1.1}$$
$$\sigma(x) = \Gamma(x)\sigma_+ + (1 - \Gamma(x))\sigma_- = F(\Gamma(x))$$

Here $\Gamma(x)$ is the characteristic function of the region V_+, equal to unity on V_+ and equal to zero on V_- and describing the structure of the composite. The local electrical conductivity $\sigma(x)$ takes the values σ_+ and σ_-, respectively. The angle brackets denote the integral of the functions within them, over the region V. The lower limit of the functional on the right-hand side of Eq.(1.1) is taken over all test functions $\varphi(x)$ assuming the values of φ_1 and φ_2 on Σ_1 and Σ_2. The electric potential makes this functional a minimum and satisfies the equations

$$\operatorname{div} \mathbf{j}(x) = 0, \quad \mathbf{j}(x) = -\sigma(x)\nabla\varphi(x)$$

and also the conditions of continuity of the normal component of the current density $\mathbf{j}(x)$ and the interphase boundary.

The current J is a functional of the characteristic function $\Gamma(x)$, given implicitly by expression (1.1). The problem consists of maximizing (or minimizing) this functional over all characteristic functions with a given mean value.

Since the magnitude of the current is proportional to the potential difference, it is convenient, when solving specific problems, to study the conductivity $\bar\sigma$ instead of the current J, which is related to the current by the formula

$$J = A\bar\sigma(\varphi_2 - \varphi_1)$$

where the constant A, which depends only on the region V, is chosen so, that when $\sigma_+ = \sigma_- = \sigma$, we have $\bar\sigma = \sigma$. To find A, we must solve the problem of the current flowing through the region V filled with a homogeneous material.

Example 1. The simplest case is the case when the region V is a rectangle in which Σ_1 and Σ_2 are the opposite sides. In the problem of maximizing the conductivity $\bar\sigma$, the

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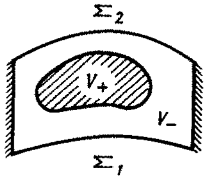

Fig.1

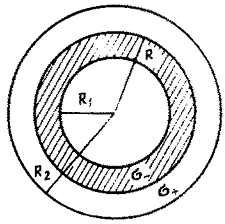

Fig.2

optimal structure will be the structure in which the materials are distributed in layers parallel to the insulated walls of the rectangle, and in the problem of minimizing $\bar{\sigma}$ the layers are perpendicular to these walls. The maximum and minimum values of the conductivity are given by the Voigt and Reuss formulas

$$\bar{\sigma}_{\max} = F(c) = c\sigma_+ + (1-c)\sigma_-, \quad \bar{\sigma}_{\min} = (c\sigma_+^{-1} + (1-c)\sigma_-^{-1})^{-1}$$

where c and $1-c$ are the proportions of the area occupied by materials with conductivity σ_+ and σ_-, respectively. We note that the solution of these problems is not unique. Any layered structure will be optimal irrespective of the number and thickness of the layers.

Example 2. In the problem of minimizing the conductivity of a ring described in polar coordinates by the relation $R_1 < r < R_2$, the optimal structure is unique (Fig.2). The inner part of the ring $R_1 < r < R$ must be filled with a poorly conducting phase, and the outer part $R < r < R_2$ with a material of high conductivity. We shall assume without loss of generality that $\sigma_+ > \sigma_-$. Then the radius R describing the interphase boundary and the minimum conductivity $\bar{\sigma}_{\min}$, will be given by the formulas

$$R^2 = cR_1^2 + (1-c)R_2^2$$
$$\bar{\sigma}_{\min} = (\sigma_-^{-1}\ln R/R_1 + \sigma_+^{-1}\ln R_2/R)^{-1}\ln R_2/R_1 \tag{1.2}$$

Proof. For the structure shown in Fig.2, the problem of electrostatics is axisymmetric and easily solved. The conductivity of this structure is found using formulas (1.2). In the course of computations we require the value of the normed multiplier $A = 2\pi(\ln R_2/R_1)^{-1}$, which is also obtained explicitly by virtue of the axial symmetry of the homogeneous problem. We require to show that for the remaining structures the conductivity is greater than the one obtained.

The radial component E_r of the electric field vector $\mathbf{E} = -\nabla\varphi$ satisfies the relation

$$\langle E_r(x)\, r^{-1}\rangle = -2\pi(\varphi_2 - \varphi_1) \tag{1.3}$$

and we can confirm this by integration by parts. The following inequality holds:

$$(\varphi_2 - \varphi_1)\, J \geqslant \inf \langle \sigma(x)\, E^2(x)\rangle \tag{1.4}$$

where the lower limit is taken on the set of all, not necessarily potential, vector fields $\mathbf{E}(x)$ satisfying the relation (1.3), i.e. in a wider class of functions than that in the variational principle (1.1). Calculation of this lower limit, after taking into account the constraint (1.3) and with help of the Lagrange multiplier, represents an algebraic problem whose solution has the form

$$\inf \langle \sigma(x)\, E^2(x)\rangle = 4\pi^2(\varphi_2 - \varphi_1)^2 \langle \sigma^{-1}(x)\, r^{-2}\rangle^{-1}$$

Let us strengthen inequality (1.4) by minimizing its right-hand side over all structures $\Gamma(x)$. To do this, we must choose the minimizing structure so as to make the quantity $\sigma(x)$ larger, at the points at which r is larger, i.e. in the outer part of the ring. This will yield the structure shown in Fig.2, where the interphase boundary is determined by the specified concentration of materials. The value of the right-hand side of inequality (1.4) is equal, for this structure, to $A\bar{\sigma}_{\min}(\varphi_2 - \varphi_1)^2$, and is larger for all the remaining structures. Thus we have obtained the lower limit for the current, and hence for the conductivity

$$(\varphi_2 - \varphi_1)\, J = A\bar{\sigma}(\varphi_2 - \varphi_1)^2 \geqslant A\bar{\sigma}_{\min}(\varphi_2 - \varphi_1)^2$$

which completes the proof.

In the general case the study of the problem in question encounters the following difficulties. Firstly, the set of characteristic functions is not linear, i.e. the functions cannot be added and multiplied by a number. This makes it impossible to use most methods of variational calculus. Secondly, there are no existence theorems for the class of problems of optimal control in question. Therefore, the upper of lower limit of the functional J may not be attainable on any structure $\Gamma(x)$, but only on some sequences of structures $\Gamma_n(x)$. For large n the structures may have an irregular, fragmented character, and in order to improve the properties of the composite we must make its structure more complex.

This situation was first discovered in /1/, and it appears to be typical in problems of optimizing the properties of composites /2, 3/. In order to obtain the optimizing sequences of the structures it is best to reformulate the problem after generalizing the concept of a composite, and after extending the domain of definition of the functional J. The generalized structures can be described by means which are more suitable for analytic applications than sequences of characteristic functions.

2. Generalized formulation of the problem. Following the ideas put forward in /1/, we shall seek the optimizing sequences of structures in the form $\Gamma_n(x) = \Gamma(x, nx)$, where the function $\Gamma(x, y)$ depends periodically on y and takes the values 0 and 1. The second argument of the function $\Gamma(x, nx)$ causes, for large values of n, rapid oscillations in the characteristic function. The structure $\Gamma_n(x)$ therefore represents, near every point x, a periodically inhomogeneous composite the geometry and periodicity cells of which change slowly from cell to cell, together with the first argument of the function $\Gamma(x, y)$. We shall call the sequences described above locally periodic generalized structures.

Near every point x, the locally periodic structure will be characterized, in particular, by the local concentrations $c(x)$ and $1 - c(x)$ of materials with conductivities σ_+ and σ_-, respectively, and an effective conductivity tensor $\sigma_{ij}(x)$. The function $c(x)$ represents a weak limit of the sequence of characteristic functions $\Gamma_n(x)$ and $\sigma_{ij}(x)$ is the G-limit of the local conductivities.

The following relation holds by virtue of the averaging theorems /4/:

$$\lim_{n\to\infty} \inf_\varphi \langle F(\Gamma_n(x))(\nabla\varphi(X))^2\rangle = \inf_\varphi \langle \sigma_{ij}(x)\nabla_i\varphi\nabla_j\varphi\rangle$$

which enables us to extend the functional J defined on the set of usual structures, to locally periodic generalized structures:

$$(\varphi_2 - \varphi_1) J = \inf_\varphi \langle \sigma_{ij}(x)\nabla_i\varphi(x)\nabla_j\varphi(x)\rangle \tag{2.1}$$

For a fixed value of the local concentration $c(x)$ at the point x, the effective conductivity tensor $\sigma_{ij}(x)$ can take various values depending on the local behaviour of the function $\Gamma(x, y)$. The set $G(c(x))$ of such values is called the accessibility set. For the class of problems discussed here, the accessibility set is described in /1/. Using the functions $c(x)$ and $\sigma_{ij}(x)$, we can restore (not uniquely) the sequences of the structures $\Gamma_n(x)$. Such sequences were produced in /1/.

When $c(x) = 0$ and $c(x) = 1$, the set $G(c(x))$ contracts to the point $\sigma_-\delta_{ij}$ or $\sigma_+\delta_{ij}$, respectively. For other values of $c(x)$ the accessibility set occupies the whole region in the space of symmetric square matrices, and the effective conductivity tensor finds a certain amount of independence. We note that the usual structures can be regarded as generalized, with the characteristic function $\Gamma_n(x)$ independent of n. The local concentration and effective conductivity tensor are identical, in the case of the usual structures, with the characteristic function $\Gamma(x)$ and the spherical tensor of local conductivities $\sigma(x)\delta_{ij}$ and the variational principle (1.1) is identical with the variational principle (2.1).

The generalized current J depends on the effective conductivity tensor $\sigma_{ij}(x)$ on the right-hand side of definition (2.1), and on the local concentration $c(x)$ which fixes the accessibility set. The generalized formulation of the optimization problem consists of maximizing (or minimizing) the current J over all functions $c(x), \sigma_{ij}(x)$ such that

$$\langle c(x)\rangle = cV,\ 0 \leqslant c(x) \leqslant 1,\ \sigma_{ij}(x) \in G(c(x)) \tag{2.2}$$

The first condition means that the proportion of the area occupied by each material is fixed and equal to c for the material with conductivity σ_+.

The generalized formulation differs from the initial formulation only in the fact that the local concentration $c(x)$ is allowed to take values different from 0 to 1. An additional optimizing parameter appears here, namely the effective conductivity tensor $\sigma_{ij}(x)$, which was initially rigidly connected with $c(x)$. Such an extension of the domain of definition makes it possible to prove the theorem of the existence of a generalized solution of the problem of maximizing (and minimizing) the current J in the class of bounded measurable functions $c(x)$, $\sigma_{ij}(x)$, over which the optimizing sequence $\Gamma_n(x)$ sought is reconstructed. The generalized solution can be non-unique.

In solving specific problems, it is useful to remember that J is a concave functional of $c(x), \sigma_{ij}(x)$, and the functional J^{-1} is concave with respect to the variables $c(x), \sigma_{ij}^{-1}(x)$. The initial formulation did not include these properties, since the functional was not defined on a linear set.

The examples given above show that amongst the solutions we may find the solution in which the local concentration $c(x)$ takes the values 0 and 1, and will represent the characteristic function of some region V_+. These solutions are not generalized and a situation may also arise in which there are no non-generalized solutions.

3. ***The problem of a ring with the lowest electrical resistance.*** Let us consider the problem of maximizing the conductivity of a ring (Fig.2), filled with materials whose conductivities are $\sigma_+, \sigma_- (\sigma_+ > \sigma_-)$. We shall seek a generalized optimal structure with local concentration $c(x)$ and effective conductivity $\sigma_{ij}(x)$. From the axial symmetry of the problem and the concavity of the functional J it follows that amongst the optimal generalized structures there exists at least one axisymmetric solution. By the axial symmetry of the generalized solution we mean the axial symmetry of the functions $c(x)$ and $\sigma_{ij}(x)$. The structures $\Gamma_n(x)$, which constitute the optimizing sequence, may themselves not have this symmetry. For the axisymmetric tensor $\sigma_{ij}(x)$ the electrical potential $\varphi(x)$ which imparts the minimum value to the right-hand side of relation (2.1), depends only on the polar radius r, and satisfies the equation $r\sigma_r(r)\varphi'(r))' = 0$ where $\sigma_r(r)$ is the radial component of the tensor $\sigma_{ij}(x)$. Having solved this equation with boundary conditions φ_1 and φ_2 on the inner and outer part of the ring, we can reduce relation (2.1) to the form

$$(\varphi_2 - \varphi_1) J = \frac{2\pi(\varphi_2 - \varphi_1)^2}{H(\sigma)}, \quad H(\sigma) = \int_{R_1}^{R_2} \frac{dr}{r\sigma_r(r)}$$

Since in the case of a ring the current and conductivity $\bar{\sigma}$ are connected by the relation $J = 2\pi(\ln R_2/R_1)^{-1}\bar{\sigma}(\varphi_2 - \varphi_1)$, it follows that

$$\bar{\sigma} = H^{-1}(\sigma)\ln R_2/R_1 \tag{3.1}$$

In order to maximize the conductivity $\bar{\sigma}$ in the class of axisymmetric generalized structures, we must minimize the functional $H(\sigma)$ over all possible local concentrations $c(r)$ and effective radial conductivities $\sigma_r(r)$. Let us fix the function $c(r)$, and minimize the functional J over the functions $\sigma_r(r)$. The effective conductivity tensor should be chosen such that its radial component $\sigma_r(r)$ is as large as possible for every value of the polar radius r. Then the functional $H(\sigma)$ will take its lowest value. The accessibility set $G(c(r))$ is constructed such that the largest possible value of the component of the effective tensor is calculated, in any one direction, using the Voigt formula and is attained on a layered structure, with the layers parallel to this direction /1/. Therefore we should put $\sigma_r(r) = F(c(r))$ in the functional $H(\sigma)$. At the points at which $0 < c(r) < 1$, the optimal generalized structure has the form of a sequence of layered structures with the layers distributed in a radial direction. Pure phases will appear at the points where $c(r) = 0$ or $c(r) = 1$.

We must now minimize the functional $H(\sigma)$ over the local concentrations $c(r)$ satisfying the condition (2.2). Without this conduction the problem would be reduced to that of minimizing the integrand in the functional $H(\sigma)$, and would produce a trivial result stating that, in order to increase the conductivity of the ring, we must fill the latter with a good conductor, provided that its supplies are unlimited.

We shall use, instead of the local concentration $c(r)$, the function $F(c(r)) = \sigma_r(r)$. In terms of this new variable the constraint (2.2), taking axial symmetry into account, will have the form

$$\int_{R_1}^{R_2} r\sigma_r(r)\,dr = F(c)\frac{R_2^2 - R_1^2}{2}, \quad \sigma_- \leqslant \sigma_r(r) \leqslant \sigma_+ \tag{3.2}$$

When minimizing the functional $H(\sigma)$, we shall use the constraint (3.2) with help of the Lagrange multiplier ω. The problem will then be reduced to that of minimizing, for every r, the expression $h = (r\sigma_r(r))^{-1} + \omega r\sigma_r(r)$. The function $\sigma_r(r)$ imparting a minimum to this expression, depends on the unknown Lagrange multiplier ω. Its substitution into the formula (3.2) yields an equation for determining ω. Expressing ω for this equation in terms of the initial parameters $R_1, R_2, \sigma_+, \sigma_-, c$ of the problem, we finally obtain the effective conductivity $\sigma_r(r)$, and hence the required generalized optimum structure. The largest conductivity of the ring $\bar{\sigma}_{max}$ is calculated for the known function $\sigma_r(r)$, using formula (3.1).

The minimum value of h can be attained at fixed r either within the interval $\sigma_- < \sigma_r(r) < \sigma_+$, or at its ends, depending on the values of the parameters r and ω. We note that the Lagrange multiplier ω must be positive, otherwise h will depend monotonically on $\sigma_r(r)$, and in order to minimize h it would be necessary to put $\sigma_r(r) = \sigma_+$, for all values of r, and this would be impossible by virtue of the constraint (3.2).

It can be confirmed that when $\omega > 0$, the minimum value of h will be imparted by the function $\sigma_r(r)$ defined by the equation

$$\sigma_r(r) = \begin{cases} \sigma_+, & r < R_+ \\ r^{-1}\omega^{-1/2}, & R_+ < r < R_- \\ \sigma_-, & r > R_- \end{cases} \tag{3.3}$$

$$R_\pm = \sigma_\pm^{-1}\omega^{-1/2}$$

In the interval $R_+ < r < R_-$ the quantity $\sigma_r(r)$ decreases monotonically from σ_+ to σ_- while the local concentration decreases from 1 to 0. The quantities R_+ and R_- determine, within the ring, the boundaries between the pure phases and the generalized layered structure.

Note that the quantities R_+ and R_- may appear outside the segment (R_1, R_2). Therefore situations are possible in which the optimal structure exists without one or both pure phases.

Let us first assume that all three phases are present in the optimal structure, i.e. that the following inequalities hold:

$$R_1 < R_+ < R_- < R_2 \tag{3.4}$$

To find the Lagrange multiplier ω, we substitute the function defined by the formula (3.3) into (3.2). This yields

$$\omega^{-1} = \sigma_+\sigma_- (cR_2^2 + (1-c) R_1^2)$$

Checking the inequalities (3.4), we obtain the domain of initial data of the problem for which the three-phase case in question is realized. The inequalities can be reduced to the form

$$\begin{gathered} c\sigma_- (R_2^2 - R_1^2) > (\sigma_+ - \sigma_-) R_1^2 \\ (1-c)\sigma_+ (R_2^2 - R_1^2) > (\sigma_+ - \sigma_-) R_2^2 \end{gathered} \tag{3.5}$$

Under such constraints imposed on the initial data the maximum conductivity $\bar{\sigma}_{max}$ is found from the formula

$$\bar{\sigma}_{max}^{-1} \ln R_2/R_1 = \sigma_-^{-1} - \sigma_+^{-1} + (\ln R_2/R_-)\sigma_-^{-1} + (\ln R_+/R_1)\sigma_+^{-1}$$

Apart from the version discussed here, three more versions of the mutual distribution of the intervals (R_1, R_2) and (R_+, R_-) are possible. In the case when the inequalities

$$R_1 < R_+ < R_2 < R_-$$

hold, the optimal structure will consist of two layers. The inner part of the ring $R_1 < r < R_+$ is filled with a material whose conductivity is σ_+, and the rest will be filled with a composite consisting of fibres finely notched in a radial direction. The radial component of the effective conductivity tensor $\sigma_r(r)$ and the quantity R_+, are calculated, as before, using formulas (3.3) in which the Lagrange multiplier ω obtained from (3.2) has the form

$$\omega^{-1} = \sigma_+^2 \{R_2 - [\sigma_+^{-1}(\sigma_+ - \sigma_-)(1-c)(R_2^2 - R_1^2)]^{1/2}\}^2$$

The domain of initial data for which this case is realized, is defined by the inequalities

$$\begin{gathered} (1-c)(\sigma_+ - \sigma_-)(R_2 + R_1) < \sigma_+ (R_2 - R_1) \\ (1-c)\sigma_+ (R_2^2 - R_1^2) < (\sigma_+ - \sigma_-) R_2^2 \end{gathered} \tag{3.6}$$

The maximum conductivity $\bar{\sigma}_{max}$ is calculated using the formula

$$\bar{\sigma}_{max}^{-1} \ln R_2/R_1 = \sigma_+^{-1} R_+^{-1} (R_2 - R_+) + \sigma_+^{-1} \ln R_+/R_1$$

When the inequalities

$$R_+ < R_1 < R_- < R_2$$

hold, the optimal structure also consists of two layers, but the inner part of the ring $R_1 < r < R_-$ is filled with the layered component, and the outer part with a poorly conducting material of conductivity σ_-. The relations describing this case have the form

$$\begin{gathered} \omega^{-1} = \sigma_-^2 \{R_1 + [\sigma_-^{-1}(\sigma_+ - \sigma_-) c (R_2^2 - R_1^2)]^{1/2}\}^2 \\ c(\sigma_+ - \sigma_-)(R_2 + R_1) < \sigma_- (R_2 - R_1) \\ c\sigma_- (R_2^2 - R_1^2) < (\sigma_+ - \sigma_-) R_1^2 \\ \bar{\sigma}_{max}^{-1} \ln R_2/R_1 = \sigma_-^{-1} R_-^{-1} (R_- - R_1) + \sigma_-^{-1} \ln R_2/R_- \end{gathered} \tag{3.7}$$

Finally, when

$$R_+ < R_1 < R_2 < R_-$$

the single-layer generalized optimal structure is characterized by the relations

$$\begin{gathered} \omega^{-1/2} = {}^1\!/_2 F(c)(R_2 + R_1) \\ c(\sigma_+ - \sigma_-)(R_2 + R_1) > \sigma_- (R_2 - R_1) \\ (1-c)(\sigma_+ - \sigma_-)(R_2 + R_1) > \sigma_+ (R_2 - R_1) \\ \bar{\sigma}_{max} = {}^1\!/_2 F(c)(R_2 + R_1)(R_2 - R_1)^{-1} \ln R_2/R_1 \end{gathered} \tag{3.8}$$

The domains of initial data of the problem given by the inequalities (3.5)-(3.8), do not intersect each other and exhaust all possible versions.

Thus the optimizing sequences of structures have been constructed for any value of

$R_1, R_2, \sigma_+, \sigma_-, c$. We note that the generalized optimal structures obtained are not unique. Functions $c(x)$ and $\sigma_{ij}(x)$ can be found which have no axial symmetry, but nevertheless produce the same maximum conductivity of the ring. It can also be shown that the problem discussed here has no non-generalized optimal structures.

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ON AN INTEGRAL EQUATION FOR AXIALLY-SYMMETRIC PROBLEMS IN THE CASE OF AN ELASTIC BODY CONTAINING AN INCLUSION*

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An approximate solution of the singular integral equation (SIE) which arises in spatial problems in the theory of elasticity with mixed conditions of one-sided detachment of inclusions under axially-symmetric torsion is considered. The singularity is taken into account using the exact solution of the equation which determines the conditions of the analogous detachment in a two-dimensional problem in the case of a sheet. It is proved that, subject to certain geometric constraints, the solution of the initial problem can be obtained by the method of successive approximations. The problem of the axially-symmetric torsion of a layer using a rigid circular disc embedded in this layer and fixed to it by one of its surfaces is treated as an example. The possiblity of the practical realization of this problem lies in the fact that a torsional moment which is applied to a rod which has been welded perpendicular to the centre of a disc can be transmitted through the disc to a layer and pierce a part of the layer.

The solution of one type of singular integral equation /1/ which arises in problems on inclusions in elastic bodies which have become detached has an integrable singularity at the ends of the integration interval. In applications such as axially-symmetric problems on detached inclusions in elastic bodies /2, 3/ the need arises to construct the solution of the above-mentioned integral equation which is bounded at one of the ends of the integration interval. This solution is constructed below by the method of "large λ" /4/.

1. Let us consider a singular integral equation in the function $q(x)$

$$\pi q(x)+\int_{-1}^{1}\frac{q(\xi)\,d\xi}{\xi-x}+\frac{1}{\lambda}\int_{-1}^{1}q(\xi)\,k\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi f(x) \tag{1.1}$$

$$(|x|\leqslant 1,\quad \lambda\in(0,\infty))$$

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